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ON ALMOST BOUNDED FUNCTIONS(1)

BY

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ABSTRACT. New results are presented with regard to the "almost bounded functions" introduced by Goodman [2], including a theorem which contains a proof of Goodman's conjecture for a particular case.

1. **Introduction.** Let E denote the unit disc $|z| < 1$. We consider the following class of functions: B is the class of functions $f(z)$ (known as the Bieberbach-Eilenberg class), regular in E such that $f(0) = 0$, and

$$f(\zeta_1)\cdot f(\zeta_2) \neq 1, \qquad \forall \zeta_1, \zeta_2 \in E. \tag{1.1}$$

$B^* \subset B$ is the subclass of univalent functions.

Let

$$G^{(2n)} = \{L_1, L_2, \ldots, L_{2n}\} \tag{1.2}$$

be a group of linear transformations where $L_j(w) = (a_j w + b_j)/(c_j w + d_j)$, $a_j d_j - b_j c_j \neq 0, j = 1, 2, \ldots, 2n$. The set of numbers generated by (1.2) for fixed w is denoted by

$$S^{(2n)}(w) = \{L_1(w), L_2(w), \ldots, L_{2n}(w)\}.$$

DEFINITION 1 (GOODMAN [2]). A function $f(z)$ is said to be "almost bounded with respect to the group $G^{(2n)}$" (A.B. for $G^{(2n)}$) in Δ if $f(z)$ is meromorphic in Δ, and if for each w (∞ included) it assumes in Δ not more than n values from the set $S^{(2n)}(w)$.

DEFINITION 1'. A point set F is said to be A.B. for $G^{(2n)}$ if, for each w, F contains at most n points of $S^{(2n)}(w)$.

If K is a linear transformation and K^{-1} its inverse, then the transformed set,

$$KG^{(2n)}K^{-1} = \{KL_jK^{-1}; j = 1, 2, \ldots, 2n\}, \tag{1.3}$$

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is again a group which may be regarded as equivalent to $G^{(2n)}$. Certain standard forms of groups of linear transformations will be considered later. In §4 we deal with functions $f(z)$, A.B. for G^*_{2n} and of the form

$$f(z) = a_1 z + a_2 z^2 + \cdots, \tag{1.4}$$

and we obtain the main result:

THEOREM 5. *Let $f(z)$ be A.B. for G^*_{2n}, of the form* (1.4) *and univalent in E; then*:

(a) $$\sum_{n=1}^{\infty} |a_n|^2 \leqslant 1,$$

(b) $$|f(z)| \leqslant \frac{|z|}{(1 - |z|^2)^{1/2}},$$

with equality for $f_r(z) = (1 - r^2)^{1/2} z/(1 + irz)$ at $z = ir$.

(c) $$|a_{n+1}| < e^{-c/2}/\sqrt{n}, \qquad n = 1, 2, \ldots,$$

where c is the Euler constant.

Part (a) of Theorem 5 obviously includes the result $|a_n| \leqslant 1, n = 1, 2, \ldots$, which solves a conjecture of Goodman [2] for a particular case.

2. The class R_{2n}.

DEFINITION 2. $\phi(z) \in R_{2n}$, $n = 1, 2, \ldots$, if:

(a) $\phi(z)$ is a regular function in E.

(b) $\phi(z)$ is A.B. for the elliptic cyclic group $G_c^{(2n)}$, defined by $G_c^{(2n)} = \{w, \eta w, \ldots, \eta^{2n-1} w\}$ where $\eta = e^{\pi i/n}$, $n = 1, 2, \ldots$.

By part (b) of Definition 2, $\phi(z) \neq 0$ in E and therefore w.l.o.g. we may assume that $\phi(z)$ has the form:

$$\phi(z) = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.1}$$

For $n = 1$ the corresponding group is $G_c^{(2)} = \{w, -w\}$ and the class R_2 coincides with the class M (first introduced by Gel'fer [4]) of regular functions which do not assume opposite values. Furthermore, we have $M \subseteq R_{2n}$ for every n, and for $n > 1$ there exist functions which belong to R_{2n} but not to M, as illustrated by the following example.

EXAMPLE 2.1. There exists, namely, a function which belongs to the class R_4 but not to M, any univalent function which maps the unit disc onto the region described in Figure 1.

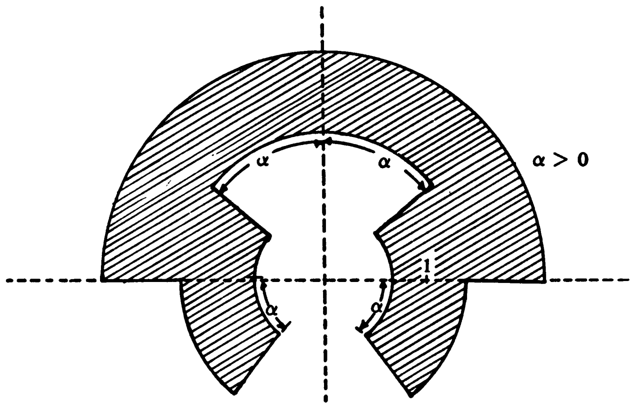

FIGURE 1: A set A.B. for R_4

We next define:

DEFINITION 3 (BIERNACKI [5, p. 94]). Let $f(z)$ be regular in an open set Δ, and $n(w)$ the number of roots in Δ of the equation $f(z) = w$. Let also:

$$p(R) = p(R, \Delta, f) = \frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\Phi})\, d\Phi, \tag{2.2}$$

$f(z)$ is called a circumferentially-mean p-valent (c. mean p-valent) function if $p(R) \leqslant p$, $0 < R < \infty$.

REMARK. It is obvious from Definition 3 that every univalent function $f(z) \in R_{2n}$, $n = 1, 2, \ldots$, is c. mean ½-valent.

THEOREM 1. *If* $\phi(z) \in R_{2n}$, $n = 1, 2, \ldots$, *and of the form* (2.1), *then*

$$|b_1| \leqslant 2. \tag{2.3}$$

If in addition $\phi(z)$ *is univalent, then*

$$\frac{1-\rho}{1+\rho} \leqslant |\phi(z)| \leqslant \frac{1+\rho}{1-\rho}, \qquad |z| = \rho,\ 0 \leqslant \rho < 1, \tag{2.4}$$

$$|\phi'(z)| \leqslant \frac{2}{1-\rho^2}|\phi(z)| \leqslant \frac{2}{(1-\rho)^2}, \qquad |z| = \rho,\ 0 \leqslant \rho < 1, \tag{2.5}$$

with equality only for the function $\phi(z) = (1 + ze^{i\theta})/(1 - ze^{i\theta})$ *for real* θ.

PROOF. As was mentioned above, every univalent function from the class R_{2n} is c. mean ½-valent. Thus using a result of Hayman [5, Theorem 5.1] we prove (2.3), (2.4), (2.5) for univalent functions. (2.3) holds also for arbitrary $\phi(z) \in R_{2n}$. This will be verified later.

We note that Theorem 1 is true for every function in the class M by the principle of subordination. This principle is inapplicable for the class R_{2n}, $n > 1$, as illustrated by the following example.

EXAMPLE 2.3. There exists a nonunivalent function $\phi(z) \in R_4$ which is not subordinate to any univalent function g in the same class R_4.

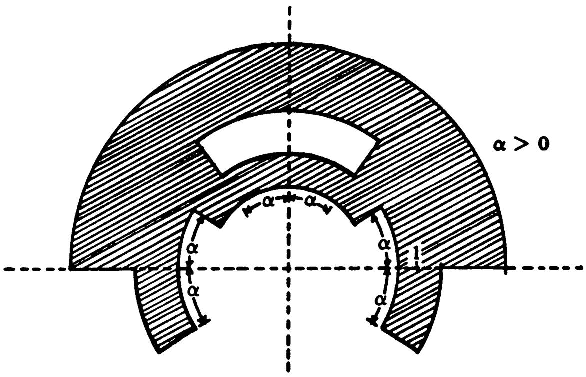

FIGURE 2: A set A.B. for $G_c^{(4)}$

To show that, we use the uniformization theorem to construct a Riemann surface conformally equivalent to the unit disc, such that its projection on the plane is the domain in Figure 2.

THEOREM 2. *Let* $\phi(z) \in R_{2n}$, $n = 1, 2, \ldots$, *be univalent and of the form* (2.1). *Denoting* $M(\rho, \phi) = \mathrm{Max}_{|z|=\rho} |\phi(z)|$, *we have*:

(a) $((1-\rho)/(1+\rho)) \cdot M(\rho, \phi)$ *is decreasing* (*as a function of* ρ, $0 < \rho < 1$, *with equality only for* $\phi(z) = (1 + ze^{i\theta})/(1 - ze^{i\theta})$, θ *a real constant*), *and thus approaches a limit* $\alpha_0 \leqslant 1$ *as* $\rho \to 1$.

(b) *The limit* $\alpha = \lim_{\rho \to 1^-} (1-\rho) M(\rho, \phi)$ *exists finitely.*

(c) *The limit* $\lim_{k\to\infty} |b_k| = \alpha/\Gamma(1) = \alpha \leqslant 2$ *exists with equality for the function* $\phi(z) = (1 + ze^{i\theta})/(1 - ze^{i\theta})$, *where* θ *is a real constant.*

(d) $$||b_{k+1}| - |b_k|| = O(k^{1-\sqrt{2}}), \quad k = 1, 2, \ldots .$$

PROOF. As $\phi(z)$ is univalent and $\phi(z) \in R_{2n}$, $\phi(z)$ is c. mean ½-valent, and (a) holds in accordance with Hayman [5, Theorem 5.1], (b) is a consequence of (a); (c) is a consequence of (b), and also correct in accordance with Hayman [5, Theorem 5.10]; (d) holds in accordance with [11].

REMARK. Using the proof procedure in [4] (see also [3]), for the class M (or R_2) we find that if $\phi(z) \in R_{2n}$, $n = 1, 2, \ldots$, and univalent, then $|b_k| < 13.56$ for $k > 1$. It seems that this estimate for the bound may be improved significantly. Moreover it is probable that ϕ is not necessarily univalent.

Goodman [2] obtained some basic results for functions which are A.B. for groups of linear transformations satisfying certain conditions. In particular, such a group is G_{2n}, obtained from $G_c^{(2n)}$ by (1.3) with $K(w) = (w + 1)/(w - 1)$:

$$G_{2n} = \left\{ L_{k+1} = \frac{(\eta^k + 1)w + (\eta^k - 1)}{(\eta^k - 1)w + (\eta^k + 1)},\ k = 0, 1, 2, \ldots, 2n - 1 \right\}; \tag{2.6}$$

$$\eta = e^{\pi i/n}.$$

A function which is A.B. for $G_2 = \{w, 1/w\}$ and of the form (1.4) belongs to the class B.

The connection which exists [4] between the classes B and M (or R_2), may be generalized to functions A.B. for G_{2n} and to those belonging to R_{2n}.

LEMMA 1. (a) *If* $\phi(z) \in R_{2n}$, $n = 1, 2, \ldots$, *and is of the form* (2.1), *then*

$$g(z) = \frac{\phi(z) - 1}{\phi(z) + 1} = \frac{b_1}{2} z + \cdots$$

is A.B. for G_{2n} *and of the form* (1.4).

(b) *If* $g(z)$, *of the form* (1.4), *is A.B. for* G_{2n}, $n = 1, 2, \ldots$, *then the function* $\phi(z)$ *defined by*

$$\phi(z) = \frac{1 + g(z)}{1 - g(z)} = 1 + b_1 z + \cdots$$

belongs to R_{2n} *and is of the form* (2.1).

Lemma 1 is a corollary of Goodman's Lemma 9 [2]. In conjunction with the result of Lai Wan-Tzei [7] it proves (2.3), and some of Goodman's results for functions A.B. for G_{2n} [2, Theorems 3 and 5] are obtained through it from Theorem 1, on a different basis.

3. **The class R_2.** Theorem 1 for the class R_2 is known [4], [6] but the proof is different. Theorem 2 was also proved for it [3], on a different basis. We now prove, for the same class,

THEOREM 3. *Let* $\phi(z) \in R_2$ *and* γ *be a real number. Assume* $\operatorname{Re}\{e^{i\gamma}\phi(z)\} > 0$, $|\gamma| < \pi/2$; *then*

$$|b_n| \leqslant 2 \cos \gamma \tag{3.1}$$

with equality for $\phi_\gamma(z) = (1 + cz)/(1 - z)$, $c = e^{2i\gamma}$, *which maps the unit disc onto the right half-plane forming an angle* γ *with the imaginary axis.*

PROOF. The case of equality is obvious. Using Cauchy's integral formula we obtain:

$$b_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{\phi(z)}{z^{n+1}}\,dz. \tag{3.2}$$

As $\phi(z)$ is regular, for all $n \geqslant 1$ and every γ,

$$0 = \frac{e^{i\gamma}}{2\pi r^n}\int_0^{2\pi} \phi(re^{i\theta})e^{ni\theta}\,d\theta.$$

Hence

$$0 = \frac{1}{2\pi r^n}\int_0^{2\pi} \overline{e^{i\gamma}\phi(re^{i\theta})e^{ni\theta}}\,d\theta. \tag{3.3}$$

By (3.2) and (3.3):

$$e^{i\gamma}b_n = \frac{1}{\pi r^n}\int_0^{2\pi} \operatorname{Re}\{e^{i\gamma}\phi(re^{i\theta})\}e^{-ni\theta}\,d\theta. \tag{3.4}$$

There exists a γ for which $\operatorname{Re}\{e^{i\gamma}\phi(re^{i\theta})\} > 0$, and therefore

$$|b_n|r^n \leqslant \frac{1}{\pi}\int_0^{2\pi} \operatorname{Re}\{e^{i\gamma}\,\phi(re^{i\theta})\}\,d\theta.$$

Using the mean-value theorem for harmonic functions and letting $r \longrightarrow 1$, we conclude that $|b_n| \leqslant 2\cos\gamma$.

REMARK. *One might conjecture that if the functions* $\phi(z) = 1 + a_1z + a_2z^2 + \cdots$, $g(z) = 1 + b_1z + b_2z^2 + \cdots$ *belong to the class* R_2, *the same is true for the function* $h(z) = \phi(z)^*g(z) = 1 + \Sigma_{n=1}^{\infty} \tfrac{1}{2}a_nb_nz^n$.

For the subclass of functions which are regular and have a positive real part in E, this conjecture is known to be true (cf. [12, Lemma 1]). The proof in [12] may be generalized for the case of a function with positive real part and another function maps the unit disc on a domain contained in a half-plane forming an angle γ with the imaginary axis. If $\phi(z)$ is regular with positive real part, $\phi_1(z) = \overline{\phi(\bar{z})}$ has the same property and, therefore: If $|k| = 1$ and $0 \leqslant \rho < 1$ then

$$0 < \operatorname{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{i\gamma}g(\rho ke^{i\theta})\operatorname{Re}[\phi_1(z)]\,d\theta\right\}$$

$$= \operatorname{Re}\left\{\frac{1}{4\pi}\int_0^{2\pi} e^{i\gamma}g(\rho ke^{i\theta})\,[\phi_1(\rho e^{i\theta}) + \overline{\phi_1(\rho e^{i\theta})}]\,d\theta\right\}$$

$$= \operatorname{Re}\left\{e^{i\gamma}\left(1 + \frac{1}{2}\sum_{\nu=1}^{\infty} a_\nu b_\nu(\rho^2k)^\nu\right)\right\}.$$

Since ρ^2k may represent any point in the unit disc, the conjecture is proved in this case. This conclusion confirms Lavie's conjecture that $|b_k| \leqslant 2$ [6], because if $\phi(z) \in R_2$ then also $\overline{\phi(\bar{z})} \in R_2$; therefore

$$h(z) = \phi(z)^*\overline{\phi(\bar{z})} = 1 + \sum_{n=1}^{\infty} \tfrac{1}{2} a_n \overline{a_n} z^n = 1 + \sum_{n=1}^{\infty} b_n z^n$$

has real coefficients and belongs to R_2, and for such functions Lavie's conjecture is known to be correct [6]. However the following example, introduced by Goodman, shows that it is not true in general; let $\phi_\alpha(z) = (1 + e^{2i\alpha}z)/(1-z)$ and $\phi_\beta(z) = (1 + e^{2i\beta}z)/(1-z)$, α and β are real constants such that $|\alpha|, |\beta| < \pi/2$ and $\sin\alpha \cdot \sin\beta > 0$, then it is not difficult to see that $H_{\alpha\beta}(z) = \phi_\alpha(z) * \phi_\beta(z)/R_2$.

4. **Functions which are A.B. for G_{2n}^*.** We now deal with functions which are A.B. for the group G_{2n}^*, defined by:

(4.1) $G_{2n}^* = \{L_{k+1} = \nu^k w,\ L_{n+k+1} = 1/\nu^k w,\ k = 0, 1, 2, \ldots, n-1\}$

where $\nu = e^{2\pi i/n}$, $n = 1, 2, \ldots$. We note that (as for the class B) a function which is A.B. for G_{2n}^* and of the form (1.4) is bounded.

The following is a key theorem for all the results for functions A.B. for G_{2n}^*.

THEOREM 4. *Let $f(z)$ be a regular function of the form* (1.4), *univalent and A.B. for G_{2n}^* in E. Let $s(f(z))$ be the area of the image of E under the mapping $f(z)$, and $\sigma(1/f(z))$ that of the complement of the image of E under the mapping $1/f(z)$. Then $s(f(z)) \leqslant \sigma(1/f(z))$.*

PROOF. If $f(z)$ is a regular function in E and of the form (1.4), there exists a disc with center at $w = 0$ and radius $r_0 < 1$, lying in the image of E under $f(z)$. As $f(z)$ is A.B. for G_{2n}^* if $|w| < r_0$ is contained within the image of E under $f(z)$, it follows that $|w| > 1/r_0$ is contained within the complement of the image.

Therefore if $f(z)$ is univalent and A.B. for G_{2n}^* in E, there exists $r_0 < 1$ such that:

(4.2) $s[\{(w = f(z)) \cap \{|w| \leqslant r_0\}\} \cup \{(w = f(z)) \cap \{|w| \geqslant 1/r_0\}\}] = \pi r_0^2,$

and

(4.3) $\sigma[\{(w = 1/f(z)) \cap \{|w| \leqslant r_0\}\} \cup \{(w = 1/f(z)) \cap \{|w| \geqslant 1/r_0\}\}] = \pi r_0^2.$

By (4.2), we have:

$$s(f(z)) - \pi r_0^2 = \pi\int_{r_0}^{1/r_0} p(\rho)\, d(\rho^2) = 2\pi\int_{r_0}^{1} p(\rho)\rho\, d\rho + 2\pi\int_{1}^{1/r_0} p(\rho)\rho\, d\rho$$

where $p(\rho)$ is defined in (2.2) and Δ is E.

Changing the integration variable, we obtain

$$s(f(z)) - \pi r_0^2 = 2\pi \int_{r_0}^{1} p(\rho)\rho\, d\rho - 2\pi \int_{1/r_0}^{1} \rho(\rho)\rho\, d\rho$$
$$= 2\pi \int_{r_0}^{1} [p(\rho)\rho + p(1/\rho)/\rho^3]\, d\rho.$$

Let us consider the pair of circles (in the image plane) $|w| = r$ and $|w| = 1/r$, where $r_0 < r \leqslant 1$.

The total length of the curves of $f(z)$ on these circles is $2\pi r p(r) + (2\pi/r)\, p(1/r)$. It is easy to see that $p(r, \Delta, 1/f) = p(1/r, \Delta, f)$, hence the total length of the complement with respect to the whole circles $|w| = r$ and $|w| = 1/r$ is:

$$2\pi r[1 - p(1/r)] + (2\pi/r)[1 - p(r)].$$

By a similar argument, we obtain:

$$\sigma(1/f(z)) - \pi r_0^2 = 2\pi \int_{r_0}^{1} \{[(1 - p(1/\rho))\rho] + [(1 - p(\rho))/\rho^3]\}\, d\rho.$$

In order to prove our theorem, we have to show that:

$$2\pi \int_{r_0}^{1} \{p(\rho)\rho + p(1/\rho)/\rho^3\}\, d\rho \leqslant 2\pi \int_{0}^{1} \{[1 - p(1/\rho)]\rho + [1 - p(\rho)]/\rho^3\} d\rho$$

or

$$0 \leqslant 2\pi \int_0^1 [1 - p(1/\rho) - p(\rho)]\,(\rho + 1/\rho^3)\, d\rho.$$

To complete the proof, we have to show that $p(\rho) + p(1/\rho) \leqslant 1$, $r_0 < \rho \leqslant 1$. As $f(z)$ is A.B. for G_{2n}^* it follows that, for each w, $f(z)$ assumes in E not more than n values from the set:

$$\left\{ w, \eta w, \ldots, \eta^{n-1} w, \frac{1}{w}, \frac{1}{\eta w}, \ldots, \frac{1}{\eta^{n-1} w} \right\}$$

where $\eta = e^{2\pi i/n}$, $n = 1, 2, \ldots$.

$f(z)$ is univalent and assumes p-values from the set $\{w, \eta w, \ldots, \eta^{n-1} w\}$ and q values from the set $\{1/w, 1/\eta w, \ldots, 1/\eta^{n-1} w\}$ and therefore $p + q \leqslant n$. Since this is true for every w, it follows that:

$$p(\rho) + p(1/\rho) \leqslant 1.$$

REMARK. Theorem 4 is obvious for the class B^*; since $f(z)$ has no values in common with $1/f(z)$ ($f(z) \neq 1/f(\zeta)$; $z, \zeta \in E$), it follows that $\{w: w = f(z)\} \subseteq C\{w: w = 1/f(z)\}$ and the inequality between the areas is obvious. If $f(z)$ is A.B. for G_{2n}^*, $f(z)$ may have common values with $1/f(z)$.

We now need the following:

LEMMA 2. *Let $f(z)$ be a regular function, univalent and A.B. for G_{2n}^* in E. Then the function $G(z) = (f(z^p))^{1/p}$, where $p > 1$ is natural, is univalent and A.B. for G_{2pn}^*.*

PROOF. Univalence is obvious. Suppose $G(z)$ assumes more than $p \cdot n$

values from the set $S^*_{2pn}(w)$; then $[G(z)]^p = f(z^p) = f(\zeta)$ assumes more than n values from $S^*_{2n}(w^p)$, which is a contradiction since $f(z)$ is A.B. for G^*_{2n}.

The following lemma is known for the class B. We generalize Grinšpan's proof [3], for our case.

LEMMA 3. *Let $f(z)$ be univalent, A.B. for G^*_{2n}, and of the form* (1.4) *in E. Denoting* $\log(f(z)/za_1) = \Sigma_{k=1}^{\infty}\beta_k z^k$, *then* $\Sigma_{k=1}^{\infty} k|\beta_k|^2 \leq \log(1/|a_1|^2)$.

PROOF. Given $f(z)$ A.B. for G^*_{2n} and univalent, we define $G(z) = (f(z^p))^{1/p}$, $p = 2, 3, \ldots$. By Lemma 2, $G(z)$ is univalent and A.B. for G^*_{2m} where $m = p \cdot n$; hence by Theorem 3, $s(G(z)) \leq \sigma(1/G(z))$. The rest of the proof is as in [3].

PROOF OF THEOREM 5. Aharonov's proof [1] for the class B may be used here, in conjunction with Lemma 3 and the inequality in [10].

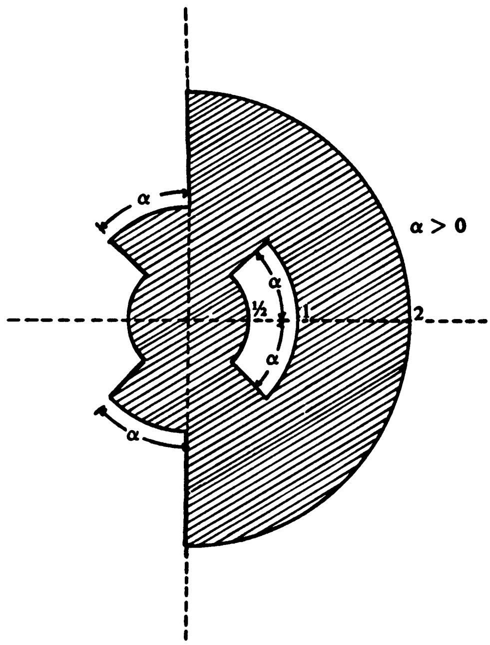

FIGURE 3: A set A.B. for G^*_8

REMARKS. (a) Lebedev and Milin [9] proved that if $f(z) \in B$ is of the form (1.4), then $|a_n| \leq 1$, $n = 1, 2, \ldots$, with equality only for $f(z) = \eta z^n$; $|\eta| = 1$, $n = 1, 2, \ldots$. The inequality (a) of Theorem 5 was first proved by Lebedev in [8], for functions having no common values.

(b) For the class B, (a) and (b) of Theorem 5 hold without the condition of univalence, and the proof for the general case is based on the fact that for every $f(z) \in B$ there exists $f^*(z) \in B^*$ such $f(z) \prec f^*(z)$. This method is inapplicable for functions A.B. for G_{2n}^* as is seen from the following example.

EXAMPLE 4.1. There exists a nonunivalent function $f(z)$ which is A.B. for G_8^* in E, but not subordinate to any univalent function belonging to G_m^* for some m.

For the set in Figure 3, we find a function which maps E on it as shown in Example 2.3.

(c) Validity of (a) of Theorem 5 implies the truth of Goodman's conjecture [2] for the group G_{2n}^*.

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